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LETTER TO THE EDITOR

Nonstandard coproducts and the Izergin–Korepin open spin chain

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Abstract. Corresponding to the Izergin–Korepin ($A_2^{(2)}$) R matrix, there are three diagonal solutions ('K matrices') of the boundary Yang–Baxter equation. Using these R and K matrices, one can construct transfer matrices for open integrable quantum spin chains. The transfer matrix corresponding to the identity matrix $K = \mathbb{I}$ is known to have $U_q(o(3))$ symmetry. We argue here that the transfer matrices corresponding to the other two K matrices also have $U_q(o(3))$ symmetry, but with a nonstandard coproduct. We briefly explore some of the consequences of this symmetry.

1. Introduction and summary

The notion of coproduct is of fundamental importance in the theory of representations of algebras. Given a representation of an algebra on a vector space V, the coproduct Δ determines the representation on the tensor product space $V \otimes V$. For a classical Lie algebra, the coproduct is trivial: for any generator x, the coproduct is $\Delta(x) = x \otimes \mathbb{I} + \mathbb{I} \otimes x$, where $\mathbb{I}$ is the identity matrix. For quantum algebras, the situation is more interesting. Indeed, consider the case $U_q(su(2))$, with a set of three generators $\{j_\pm, h\}$ obeying

$$[h, j_\pm] = \pm j_\pm. \tag{1}$$

As is well known, the 'standard' coproduct

$$\begin{aligned} \Delta(h) &= h \otimes \mathbb{I} + \mathbb{I} \otimes h \\ \Delta(j_\pm) &= j_\pm \otimes q^h + q^{-h} \otimes j_\pm \end{aligned} \tag{2}$$

is compatible with the commutation relation

$$[j_+, j_-] = \frac{q^{2h} - q^{-2h}}{q - q^{-1}}. \tag{3}$$

Perhaps less well known is the fact that there is also a 'nonstandard' coproduct

$$\begin{aligned} \Delta(h) &= h \otimes \mathbb{I} + \mathbb{I} \otimes h \\ \Delta(j_\pm) &= j_\pm \otimes \mathbb{I} + q^h \otimes j_\pm \end{aligned} \tag{4}$$

which is compatible instead with the q-commutation relation

$$j_+ j_- - q^{-1} j_- j_+ = \frac{\mathbb{I} - q^{2h}}{1 - q^2}. \tag{5}$$

Remarkably, both of these types of coproducts can be realized in the open integrable quantum spin chain constructed with the $A_2^{(2)}$ R matrix [1] by choosing appropriate boundary conditions. Let us briefly recall the history of this model. Sklyanin [2] pioneered the generalization of the quantum inverse scattering method (QISM) [3] to systems with boundaries, and showed that integrable boundary conditions can be obtained from solutions $K(u)$ of the boundary Yang–Baxter equation [4, 5]. This approach was then generalized [6] to spin chains associated with general affine Lie algebras [7, 8]. In particular, for the $A_2^{(2)}$ case, it was found [9] that there are only three diagonal solutions of the boundary Yang–Baxter equation:

$$
\begin{aligned}
K^{(0)}(u) &= \mathbb{I} = \mathrm{diag}(1, 1, 1) \\
K^{(1)}(u) &= \mathrm{diag}\left(\mathrm{e}^{-u}, \frac{\sinh(\frac{1}{2}(3\eta - \frac{\mathrm{i}\pi}{2} + u))}{\sinh(\frac{1}{2}(3\eta - \frac{\mathrm{i}\pi}{2} - u))}, \mathrm{e}^{u}\right) \\
K^{(2)}(u) &= \mathrm{diag}\left(\mathrm{e}^{-u}, \frac{\cosh(\frac{1}{2}(3\eta - \frac{\mathrm{i}\pi}{2} + u))}{\cosh(\frac{1}{2}(3\eta - \frac{\mathrm{i}\pi}{2} - u))}, \mathrm{e}^{u}\right)
\end{aligned}
\tag{6}
$$

where u is the spectral parameter, and η is the anisotropy parameter. Let us denote the corresponding transfer matrices for open quantum spin chains with N sites by $t^{(i)}(u)$, $i = 0, 1, 2$. (The construction of these transfer matrices is described below in section 2.) It was shown in [10, 11] that the transfer matrix $t^{(0)}(u)$ constructed with the identity matrix $K^{(0)}$ has $U_q(o(3))$ symmetry:

$$[t^{(0)}(u), S^{\pm}] = 0 \qquad [t^{(0)}(u), S^3] = 0 \tag{7}$$

where the generators obey

$$[S^3, S^{\pm}] = \pm S^{\pm} \qquad [S^+, S^-] = \frac{q^{2S^3} - q^{-2S^3}}{q - q^{-1}} \tag{8}$$

and

$$S^{\pm} = \sum_{k=1}^{N} q^{s_N^3 + \cdots + s_{k+1}^3} s_k^{\pm} q^{-(s_{k-1}^3 + \cdots + s_1^3)} \qquad S^3 = \sum_{k=1}^{N} s_k^3 \tag{9}$$

where

$$s^+ = \sqrt{2\cosh\eta}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad s^- = \sqrt{2\cosh\eta}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad s^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \tag{10}$$

and $q = \mathrm{e}^{\eta}$. That is, the transfer matrix has quantum algebra symmetry with the 'standard' coproduct (2). This is a generalization of the observation [12, 13] of $U_q(su(2))$ symmetry for the $A_1^{(1)}$ case. Batchelor and Yung [14] later showed that the open $A_2^{(2)}$ spin chain can be mapped to the problem of polymers at surfaces, and that the above three solutions $K^{(i)}(u)$ correspond to three distinct surface critical behaviours.

There has remained the question: what symmetry—if any—do the transfer matrices constructed with $K^{(1)}$ and $K^{(2)}$ have? Naively, one expects that since $K \neq \mathbb{I}$, there is less symmetry†. However, this is *not* the case. We argue here that the transfer matrices $t^{(1)}(u)$ and $t^{(2)}(u)$ also have $U_q(o(3))$ symmetry, but with a 'nonstandard' coproduct (4):

$$[t^{(i)}(u), S^{\pm}] = 0 \qquad [t^{(i)}(u), S^3] = 0 \qquad i = 1, 2 \tag{11}$$

where the generators obey

$$[S^3, S^{\pm}] = \pm 2S^{\pm} \qquad S^+S^- - q^{-2}S^-S^+ = \frac{\mathbb{I} - q^{2S^3}}{1 - q^2} \tag{12}$$

† This expectation holds true for the $A_n^{(1)}$ case [15]. Indeed, there the diagonal K matrices contain an additional continuous parameter ξ; and $K = \mathbb{I}$ is a point ($\xi \to \infty$) of enhanced symmetry.

and

$$S^{\pm} = \sum_{k=1}^{N} s_k^{\pm} q^{s_{k-1}^3 + \cdots + s_1^3} \qquad S^3 = \sum_{k=1}^{N} s_k^3 \tag{13}$$

where

$$s^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad s^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad s^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \tag{14}$$

and $q = \mathrm{e}^{4\eta}$. Knowledge of such symmetry is essential for understanding important features of the models such as degeneracies of the spectrum and the Bethe Ansatz solution. Equations (11)–(14) are the main results of this letter. In section 2 we provide some pertinent details about the construction and symmetry of the models, and we conclude in section 3 with a brief discussion.

2. Some details

In this section, we briefly review the construction of the transfer matrices, and outline the argument for their symmetry. The solution $R(u)$ of the Yang–Baxter equation found by Izergin and Korepin [1], which corresponds [7, 8] to the case $A_2^{(2)}$, can be written in the following form [11, 16]:

$$R(u) = \left(\begin{array}{ccc|ccc|ccc} c & & & & & & & & \\ & b & & e & & & & & \\ & & d & & g & & f & & \\ \hline & \bar{e} & & b & & & & & \\ & & \bar{g} & & a & & g & & \\ & & & & & b & & e & \\ \hline & & \bar{f} & & \bar{g} & & d & & \\ & & & & & \bar{e} & & b & \\ & & & & & & & & c \end{array}\right) \tag{15}$$

where

$a = \sinh(u - 3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh \eta \qquad b = \sinh(u - 3\eta) + \sinh 3\eta$

$c = \sinh(u - 5\eta) + \sinh \eta \qquad d = \sinh(u - \eta) + \sinh \eta$

$e = -2\mathrm{e}^{-\frac{u}{2}} \sinh 2\eta \cosh\left(\frac{u}{2} - 3\eta\right) \qquad \bar{e} = -2\mathrm{e}^{\frac{u}{2}} \sinh 2\eta \cosh\left(\frac{u}{2} - 3\eta\right)$

$f = -2\mathrm{e}^{-u+2\eta} \sinh \eta \sinh 2\eta - \mathrm{e}^{-\eta} \sinh 4\eta \qquad \bar{f} = 2\mathrm{e}^{u-2\eta} \sinh \eta \sinh 2\eta - \mathrm{e}^{\eta} \sinh 4\eta$

$g = 2\mathrm{e}^{-\frac{u}{2}+2\eta} \sinh \frac{u}{2} \sinh 2\eta \qquad \bar{g} = -2\mathrm{e}^{\frac{u}{2}-2\eta} \sinh \frac{u}{2} \sinh 2\eta.$

It has the regularity property $R(0) \propto \mathcal{P}$, where $\mathcal{P}$ is the permutation matrix, as well as unitarity, PT symmetry, and crossing symmetry:

$$R_{12}(u) = V_1 R_{12}(-u - \rho)^{t_2} V_1 = V_2^{t_2} R_{12}(-u - \rho)^{t_1} V_2^{t_2} \tag{16}$$

where the crossing matrix V is given by

$$V = \begin{pmatrix} & & -\mathrm{e}^{-\eta} \\ & 1 & \\ -\mathrm{e}^{\eta} & & \end{pmatrix} \tag{17}$$

and $\rho = -6\eta - \mathrm{i}\pi$.

Given a solution $K(u)$ of the boundary Yang–Baxter equation, a corresponding transfer matrix $t(u)$ for an open integrable quantum spin chain with N sites is given by [2, 6, 15]

$$t(u) = \mathrm{tr}_0\, M_0 K_0(-u-\rho)^{t_0} \mathcal{T}_0(u) \tag{18}$$

where

$$\mathcal{T}_0(u) = T_0(u) K_0(u) \hat{T}_0(u) \tag{19}$$

with

$$T_0(u) = R_{0N}(u) \dots R_{01}(u) \qquad \hat{T}_0(u) = R_{10}(u) \dots R_{N0}(u) \tag{20}$$

and

$$M = V^t V = \mathrm{diag}(\mathrm{e}^{2\eta}, 1, \mathrm{e}^{-2\eta}). \tag{21}$$

Indeed, the transfer matrix forms a one-parameter commutative family $[t(u), t(v)] = 0$, which contains the Hamiltonian $\mathcal{H}$,

$$\mathcal{H} \propto \frac{\mathrm{d}}{\mathrm{d}u} t(u)\bigg|_{u=0}. \tag{22}$$

For the three K matrices $K^{(i)}(u)$ given in equation (6), we denote by $t^{(i)}(u)$ the corresponding transfer matrices, and by $\mathcal{H}^{(i)}$ the corresponding Hamiltonians. We now restrict our attention to the cases $i = 1, 2$. For two sites ($N = 2$), we have checked the $U_q(o(3))$ symmetry, (11)–(14), of the transfer matrix by direct computation. In particular, equation(22) implies that the two-site Hamiltonian also has this symmetry. For general N, the Hamiltonian is given by a sum of two-site Hamiltonians plus boundary terms. It follows that, for general N, the Hamiltonian $\mathcal{H}^{(i)}$ has $U_q(o(3))$ symmetry

$$[\mathcal{H}^{(i)}, S^{\pm}] = 0 \qquad [\mathcal{H}^{(i)}, S^3] = 0 \qquad i = 1, 2 \tag{23}$$

where the symmetry generators obey (12)–(14). We have also checked the symmetry (11) of the transfer matrix for three sites ($N = 3$) by direct computation, and we conjecture that it holds for general N.

We remark that the symmetry generators $S^{\pm}$, S^3 defined in (13), (14) lie in the fundamental algebraic structures of QISM. Indeed, note the asymptotic behaviour of the R and K matrices for $u \to \infty$:

$$R(u) \sim \mathrm{e}^u R^+ + R^{++} + \mathrm{O}(\mathrm{e}^{-u}) \tag{24}$$

$$K^{(i)}(u) \sim \mathrm{e}^u K^{(i)+} + K^{(i)++} + \mathrm{O}(\mathrm{e}^{-u}) \qquad i = 1, 2 \tag{25}$$

where R^+, R^{++}, $K^{(i)+}$, $K^{(i)++}$ are independent of u. It follows that the quantity $\mathcal{T}^{(i)}(u)$ defined as in equation (19) has the asymptotic behaviour for $u \to \infty$

$$\mathcal{T}^{(i)}(u) \sim \mathrm{e}^{(2N+1)u} \mathcal{T}^{(i)+} + \mathrm{e}^{2Nu} \mathcal{T}^{(i)++} + \cdots \tag{26}$$

where $\mathcal{T}^{(i)+}$, $\mathcal{T}^{(i)++}$ are independent of u. The basic observation is that the generators $S^{\pm}$ lie in the antidiagonal corners of $\mathcal{T}^{(i)++}$ (viewed as a 3×3 auxiliary-space matrix, with operator-valued entries):

$$\mathcal{T}^{(i)++} = \begin{pmatrix} 0 & 0 & S^- \\ 0 & * & * \\ S^+ & * & * \end{pmatrix}. \tag{27}$$

We expect that this observation will be useful for formulating a QISM proof of the symmetry (11).

3. Discussion

One immediate consequence of the symmetry which we have uncovered is the explanation of degeneracies in the spectrum for finite N. For instance, consider the pseudovacuum vector $\omega = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^{\otimes N}$,

$$t^{(i)}(u)\omega = \Lambda^{(i)}(u)\omega \qquad i = 1, 2 \tag{28}$$

where $\Lambda^{(i)}(u)$ is the corresponding pseudovacuum eigenvalue. Commutativity of the transfer matrix with S^- implies that the vectors $(S^-)^n\omega$ for $n = 1, 2, \ldots, N$ are also eigenvectors of the transfer matrix with the same eigenvalue. Moreover, we observe that each site carries a *reducible* representation of the $U_q(o(3))$ algebra, namely $\mathbf{2} \oplus \mathbf{1}$ (instead of $\mathbf{3}$), implying the degeneracy pattern $(\mathbf{2} \oplus \mathbf{1})^{\otimes N}$.

Note that the pseudovacuum vector ω is annihilated by S^+; that is, $S^+\omega = 0$. We expect that all Bethe Ansatz states (which can presumably be constructed by applying appropriate creation-like operators to ω) are such highest-weight states. (See, e.g., [10, 17–19].)

Finally, we remark that we have considered here only the first of the infinite family of models $A^{(2)}_{2n}$, $n = 1, 2, \ldots$. For these R matrices [7, 8], there are again only three distinct diagonal solutions of the boundary Yang–Baxter equation: $K^{(0)} = \mathbb{I}$ [9], and $K^{(1)}$, $K^{(2)}$ given in [20]. The transfer matrix constructed with $K^{(0)}$ has [10] the symmetry $U_q(o(2n+1))$ with the standard coproduct. We expect that the transfer matrices constructed with $K^{(1)}$ and $K^{(2)}$ also have $U_q(o(2n+1))$ symmetry, but with a nonstandard coproduct. We hope to report on this and related matters in a future publication.

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